Ladder Operator Review

Simple Harmonic Oscilator

Lingo

$$\psi_n = |n\rangle = \begin{pmatrix} c1 \\ c2 \\ c3 \\ \vdots \end{pmatrix} \qquad \text{Ground state} = |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$1\text{ st excited state} = |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad 2\text{ nd excited state} = |2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

The ladder opperators a and a^+

$$\text{lowering operator} = \hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\,\omega}{\hbar}}\,\hat{x} + \frac{i}{\sqrt{m\,\omega\,\hbar}}\,\hat{P}\right)$$

$$= \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \searrow & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{n} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \Box \end{pmatrix}$$

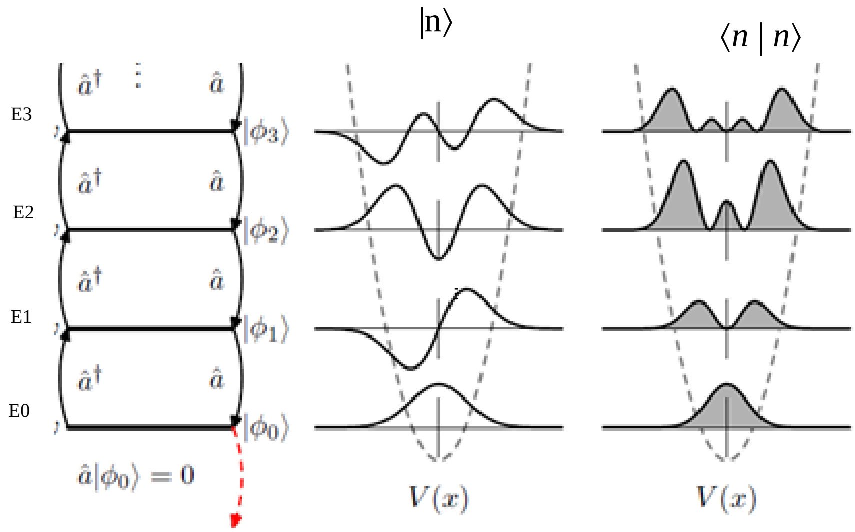
$$\text{raising operator} = \hat{a}^+ = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} - \frac{i}{\sqrt{m\omega\hbar}}\,\hat{P}\right)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \searrow & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{n+1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \Box \end{pmatrix}$$

$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ where $n > 0$ since you can't go lower than the ground state

$\hat{a}^+\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

<u>Ladder operators can be used to find the excited states in a SHO</u>



http : // personal.ph.surrey.ac.uk/~ja0017/chapter2.pdf

Lets find the first excited state in a SHO using two methods

$$\hat{a}\,|2\rangle = \sqrt{2}\,|2-1\rangle \qquad \hat{a}^{+}\,|0\rangle = \sqrt{0+1}\,|0+1\rangle$$

$$= \sqrt{2}\,|1\rangle \qquad = \sqrt{1}\,|1\rangle$$

$$= |1\rangle$$

$$\hat{a}\,|2\rangle = \begin{pmatrix} 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \\ \vdots \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \sqrt{2}\,|1\rangle$$

<u>The number operator:</u> $\hat{N} = a^{+}\,a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 2 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \searrow & 0 & \ldots \\ 0 & 0 & 0 & 0 & n & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \square \end{pmatrix}$

$$\hat{N}\,|\,n\rangle = \hat{a}^{+}\,\hat{a}\,|\,n\rangle$$

$$= \hat{a}^{+}\,\sqrt{n}\,|\,n-1\rangle$$

$$= \sqrt{n}\,\hat{a}^{+}\,|\,n-1\rangle$$

$$= \sqrt{n}\,\sqrt{(n-1)+1}\,|\,(n-1)+1\rangle$$

$= \; n \,|\, n\rangle$

Hamiltonian

$$H = \hbar\,\omega\left(\hat{a}^{+}\,\hat{a} + \frac{1}{2}\right) \qquad\qquad H\left|1\right\rangle = \hbar\,\omega\left(\hat{N} + \frac{1}{2}\right)\left|1\right\rangle$$

$$= \hbar\,\omega\left(\hat{N} + \frac{1}{2}\right) \qquad\qquad = \hbar\,\omega\,(1\,|\,1\rangle + \frac{1}{2}\left|1\right\rangle)$$

$$= \hbar\,\omega\left(\frac{3}{2}\right)\left|1\right\rangle$$

$$E1 = \hbar\,\omega\left(\frac{3}{2}\right)$$

Useful relations

$$\hat{a}\,|\,0\rangle = 0$$

$$\left[\hat{a}, \hat{a}^{+}\right] = 1 \qquad \left[H, \hat{a}^{+}\right] = \hbar\,\omega\,\hat{a}^{+} \qquad \left[H, \hat{a}^{+}\right] = -\hbar\,\omega\,\hat{a}$$

Angular momentum

First off, we know J=L+S. Which one you use depends on your system. I'm most used to seeing it expressed with L, but the same relations hold for the other values.

The raising and lowering operators are expresses interms of the direstional angular momentum operators shown here in cartesian. They get much uglier in spherical coordinates.

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = yp_z - zp_y,$$

$$L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) = zp_x - xp_z,$$

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = xp_y - yp_x.$$

$$\begin{aligned}[L_x, L_y] &= [yp_z - zp_y, zp_x - xp_z],\\ &= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z],\\ &= y[p_z, z]p_x - x[p_z, z]p_y,\\ &= -i\hbar(yp_x - xp_y),\\ &= i\hbar L_z.\end{aligned}$$

http : // ursula.chem.yale.edu/~batista/vvv/node16.html

$L_{\pm} = L_x \pm i\,L_y$

$\Longleftarrow$ Definition of Ladder Operators for angular momentum

Finding eigenvalues of angular momentum ladder operators

$$L_{+}\,|\,l, m_l\rangle = c^{+}_{l,m_l}\,\hbar\left|\,l, m_l + 1\right\rangle$$

$$\langle l, m_l + 1\,|\,L_{+}\,|\,l, m_l\rangle = c^{+}_{l,m_l}\,\hbar$$

c^{+}_{l,m_l} is a normalization constant that we find by comparing two ways of solving $L_- L_+ \left| l, m_l \right\rangle$

$$L_- L_+ | l, m_l\rangle = L_- c^{+}_{l,m_l} \hbar \left| l, m_l \right\rangle$$
$$= c^{+}_{l,m_l} \hbar L_- \left| l, m_l \right\rangle$$
$$= c^{+}_{l,m_l} \hbar c^{-}_{l,m_l} \hbar \left| l, m_l \right\rangle$$
$$\Downarrow \quad \left(c^{+}_{l,m_l}\right)^* = c^{-}_{l,m_l}$$
$$= \left| c^{+}_{l,m_l} \right|^2 \hbar^2 \left| l, m_l \right\rangle$$

$$L_- L_+ | l, m_l\rangle = \left(L_x - i L_y\right)\left(L_x + i L_y\right) \left| l, m_l \right\rangle$$
$$= \left(L_x{}^2 + L_y{}^2 - i\left[L_x, L_y\right]\right) \left| l, m_l \right\rangle$$
$$\Downarrow \quad \left[L_x, L_y\right] = i\hbar L_z$$
$$= \left(L^2 - L_z{}^2 - \hbar L_z\right) \left| l, m_l \right\rangle$$
$$= L^2 \left| l, m_l \right\rangle - L_z{}^2 \left| l, m_l \right\rangle - \hbar L_z \left| l, m_l \right\rangle$$
$$\Downarrow \quad L^2 \left| l, m_l \right\rangle = \hbar^2 l (l+1) \left| l, m_l \right\rangle$$
$$\Downarrow \quad L_z | l, m_l\rangle = m_l \hbar | l, m_l\rangle$$
$$= \hbar^2 (l(l+1) - m_l (m_l + 1)) \left| l, m_l \right\rangle$$

Now we can set the two solutions equal and solve for the normalization constant

$$\left| c^{+}_{l,m_l} \right|^2 \hbar^2 = \hbar^2 (l(l+1) - m_l (m_l + 1))$$
$$c^{+}_{l,m_l} = \sqrt{(l(l+1) - m_l (m_l + 1))}$$

the same process can be followed for c^{-}_{l,m_l}

$$c^{-}_{l,m_l} = \sqrt{(l(l+1) - m_l (m_l - 1))}$$

$$L_\pm | l, m_l\rangle = \sqrt{l(l+1) - m_l (m_l \pm 1)} \, \hbar \left| l, m_l \pm 1 \right\rangle$$

<u>Ladder operators are good for solving matrix elements</u>

$$\langle l, m_l + 1 | L_x | l, m_l\rangle =$$
$$\Downarrow \; 2 L_x = \left(L_x + i L_y\right) + \left(L_x - i L_y\right) = L_- + L_+$$
$$\Downarrow \; L_x = \frac{L_- + L_+}{2}$$
$$= \left\langle l, m_l + 1 \left| \frac{L_-}{2} \right| l, m_l \right\rangle + \left\langle l, m_l + 1 \left| \frac{L_+}{2} \right| l, m_l \right\rangle$$

$$= \left\langle l, m_l + 1 \right| \frac{\sqrt{l(l+1) - m_l(m_l - 1)}\ \hbar}{2} \left| l, m_l - 1 \right\rangle$$

$$+ \left\langle l, m_l + 1 \right| \frac{\sqrt{l(l+1) - m_l(m_l + 1)}\ \hbar}{2} \left| l, m_l + 1 \right\rangle$$

$$= 0 + \left\langle l, m_l + 1 \right| \frac{\sqrt{l(l+1) - m_l(m_l + 1)}\ \hbar}{2} \left| l, m_l + 1 \right\rangle$$

$$= \frac{\sqrt{l(l+1) - m_l(m_l + 1)}\ \hbar}{2} \langle l, m_l + 1 \mid l, m_l + 1 \rangle$$

$$= \frac{\sqrt{l(l+1) - m_l(m_l + 1)}\ \hbar}{2}$$

Challenge problem : $\left\langle l, m_l + 1 \right| L_x^3 \left| l, m_l \right\rangle$

Useful relations

$[L_z, L_\pm] = \pm\hbar\ L_\pm \qquad [L_+, L_-] = 2\,\hbar\ L_z$

$[L_z, L_x] = i\,\hbar\ L_y \qquad [L_x, L_y] = i\,\hbar\ L_z \qquad [L_y, L_z] = i\,\hbar\ L_x$

$L^2 = L_x^{\,2} + L_y^{\,2} + L_z^{\,2} \qquad [L^2, L] = 0$

$L^2 \left| l, m_l \right\rangle = \hbar^2\, l(l+1) \left| l, m_l \right\rangle$ when $l \geq 0$

$L_z \,|\, l, m_l\rangle = m_l\,\hbar \,|\, l, m_l\rangle$ when $-l \leq m_l \leq l$

For a Spin = 1 system

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \qquad L^2 = 2\,\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$