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Electric Fields Produced by the Propagation of Plane Coherent Electromagnetic Radiation in a Stratified Medium

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Explicit formulas are derived for the mean-square electric fields induced by plane electromagnetic radiation in a two-phase, three-phase, and N-phase stratified medium. The first (incident) and last phases are semi-infinite in extent. Boundaries separating phases are plane and parallel. Phases are isotropic with arbitrary optical constants. Simple relationships follow for special cases such as at the critical angle for a two-phase system. Equations for reflectance, transmittance, and phase changes on reflectance and transmittance are given. Details are given concerning the energy absorption process, especially in the two- and three-layer cases. Equations for the N-layer case are in terms of characteristic matrices which can be readily programmed for a computer.

INDEX HEADINGS: Reflectance; Transmittance; Absorption; Coherence; Spectroscopy.

REFLECTION spectroscopy deals with the reflection of electromagnetic radiation from one or more phases, often in the form of a stratified medium of several layers.[1,2] In general, the phases are absorbing—even metallic. To understand the mechanism of absorption, and, in particular, to derive the spectrum of a certain phase from the observed reflection spectrum, it is imperative that the magnitude and direction of the electric fields due to the radiation be known. The main purpose of this paper is to derive explicit expressions for the electric fields arising from coherent radiation in a stratified medium comprising homogeneous layers with parallel plane boundaries.

Reflection from a three-phase system is considered first. Explicit expressions are given for reflectance, transmittance, and phase changes. The dominant role played by the mean-square electric fields is then described, and the fields are derived. The derivation is accomplished in a way to give physical insight into this important three-phase case, and the equations for the fields are explicitly expressed in terms of the reflectance or transmittance and the phase changes.

The simpler two-phase case is discussed separately because of its importance and because simple equations of great utility can be easily derived from the more general theory.

Finally, a system of N phases is considered, i.e., N-2 parallel bounded layers between semi-infinite initial and final phases. In this case the reflectance, etc., and fields are given in terms of matrices characteristic of the stratified medium. The physics of the interaction is not as apparent from these equations as it is in less general cases, but the equations are explicit and in a form easily programmed on a computer.

DERIVATION OF EQUATIONS

A. Three-Phase System

We consider here the general case of plane-wave radiation interacting with a three-phase system, as shown in Fig. 1. The phases are isotropic and homogeneous with plane boundaries, as indicated. The initial phase is transparent while the optical constants of the second and third phases can have any values whatever. The optical properties of each phase are characterized completely by three constants (if the permeability μ is complex, four constants are required) which are functions of wavelength, e.g., the dielectric constant, ϵ, the magnetic permeability, μ, and the conductivity, σ. An alternative set is μ plus the complex dielectric constant $\hat{\epsilon}\equiv\epsilon+i4\pi\sigma/\omega$, where ω is the angular frequency. Still another set of constants is used in optics, μ and the complex index of refraction, $\hat{n}$, which we choose to express as $\hat{n}\equiv n+ik$, where n is the usual real index of refraction and k is called the extinction coefficient. The complex index is simply related to the previous constants, $\hat{n}^2=\mu\hat{\epsilon}$. Various other relations among the constants are given in Ref. 3, p. 610.

The coordinates are as indicated, with the y-axis perpendicular to the plane of Fig. 1. Angles of incidence or refraction are given by θ_j, where the subscript is used to indicate the phase involved. The thickness of the second phase is h, although h/λ will be used for convenience, where λ is wavelength in vacuo. It is also convenient to define $\xi_j\equiv\hat{n}_j\cos\theta_j=(\hat{n}_j^2-n_1^2\sin^2\theta_1)^{\frac{1}{2}}=\mathrm{Re}\xi_j+i\,\mathrm{Im}\xi_j$ and $\beta_j\equiv2\pi(h/\lambda)\xi_j$. It is not at all obvious which roots are to be taken to obtain the real and imaginary parts of ξ_j. It will be shown later that, if the time dependence of the field is taken as $e^{-i\omega t}$ and k is taken positive for an absorbing medium, both $\mathrm{Re}\xi_j$ and $\mathrm{Im}\xi_j$ must always be taken ≥0.

Reflectance, Transmittance, and Phase Changes Perpendicular Polarization (transverse electric, TE)

Generalized Fresnel formulas for the complex reflection and transmission coefficients at a plane boundary between two phases j and k for perpendicular po-

[1] N. J. Harrick, J. Opt. Soc. Am. 55, 851 (1965).
[2] W. N. Hansen, T. Kuwana, and R. A. Osteryoung, Anal. Chem. 38, 1810 (1966).
[3] a. M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Inc., New York, 1959), p. 61; b. p. 609.

larization are[4]

$$r_{\perp jk}=\frac{\mu_k\xi_j-\mu_j\xi_k}{\mu_k\xi_j+\mu_j\xi_k},\quad t_{\perp jk}=\frac{2\mu_k\xi_j}{\mu_k\xi_j+\mu_j\xi_k},\tag{1}$$

light being incident from phase j.

The corresponding reflection and transmission coefficients for the three-phase combination of Fig. 1 can be derived by a generalization of such coefficients for the nonabsorbing dielectric case given, for example, in Ref. 3a. The generalization consists of replacing ϵ with $\hat{\epsilon}$ (recall that $\hat{\epsilon}=\epsilon+i4\pi\sigma/\omega$) which makes n and θ_j complex, in general. The new equations are valid because in the specialized Maxwell's equations from which they are derived, absorption can be accounted for by replacing ϵ with $\hat{\epsilon}$ in the formulas for the nonabsorbing case.[3b] In a formal way then, the old solutions for the nonabsorbing case still apply. Therefore, we can write for the three-phase combination of Fig. 1,

$$r_\perp=\frac{r_{\perp 12}+r_{\perp 23}e^{2i\beta}}{1+r_{\perp 12}r_{\perp 23}e^{2i\beta}},\quad t_{E\perp}=\frac{t_{\perp 12}t_{\perp 23}e^{i\beta}}{1+r_{\perp 12}r_{\perp 23}e^{2i\beta}}.\tag{2}$$

The coefficient r refers to the reflected light in phase 1 while $t_{E\perp}$ is the ratio of the electric-field amplitude in phase 3 at the boundary to the field amplitude of the incident beam in phase 1. Note that these equations retain this same form for widely divergent conditions, including the case of strongly absorbing phases 2 and 3, or total internal reflection at either boundary. The reflectance and transmittance of the three-phase system of Fig. 1 are

$$R_\perp=|r_\perp|^2,\quad T_\perp=\frac{\mu_1\,\mathrm{Re}\xi_3}{\mu_3\xi_1}|t_{E\perp}|^2.\tag{3}$$

The transmittance is the fraction of the energy incident on boundary 1,2 that ends up in phase 3. If we define the phase change on reflection, $\delta_\perp^r$, as the argument of the complex reflection coefficient, $r_\perp$, and $\delta_\perp^t$ for the corresponding phase change in transmission, we can write

$$\delta_\perp^r=\arg r_\perp,\quad\text{and}\quad \delta_\perp^t=\arg t_{E\perp}.\tag{4}$$

It is important to keep in mind that the δ's may have values relating to any quadrant of the complex plane. Recall that the argument of a complex number z is given by $\arg z=\tan^{-1}(\mathrm{Im}\,z/\mathrm{Re}\,z)$ for $\mathrm{Re}\,z>0$ and $\arg z=\tan^{-1}(\mathrm{Im}\,z/\mathrm{Re}\,z)+\pi$ for $\mathrm{Re}\,z<0$. For $\mathrm{Re}\,z=0$, $\arg z=\pi/2$ for $\mathrm{Im}\,z>0$ and $\arg z=-\pi/2$ for $\mathrm{Im}\,z<0$. For $\mathrm{Re}\,z=0=\mathrm{Im}\,z$, $\arg z$ is undefined.

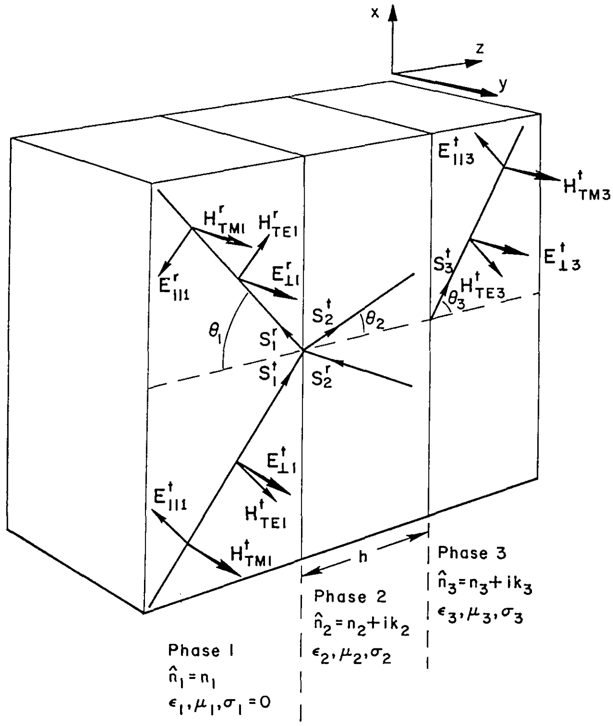

FIG. 1. Interaction of plane wave in a three-phase system. Sign conventions for parallel (‖ or TM) and perpendicular (⊥ or TE) polarized radiation are illustrated—the indicated geometry is for zero phase change, in all cases. The symbol **s** refers to unit propagation vector.

When numerical calculations are being made, especially on a computer, the form of Eqs. (3) is most convenient. For mathematical analysis, however, it may be desirable to give the results in more explicit form, especially for those who prefer not to think in terms of complex algebra. When multiplied out, the reflectance is

$$R_\perp=\frac{R_{\perp 12}+R_{\perp 23}e^{-4\,\mathrm{Im}\beta}+R_{\perp 12}^{1/2}R_{\perp 23}^{1/2}e^{-2\,\mathrm{Im}\beta}2\cos(\delta_{\perp 23}^r-\delta_{\perp 12}^r+2\,\mathrm{Re}\beta)}{1+R_{\perp 12}R_{\perp 23}e^{-4\,\mathrm{Im}\beta}+R_{\perp 12}^{1/2}R_{\perp 23}^{1/2}e^{-2\,\mathrm{Im}\beta}2\cos(\delta_{\perp 23}^r+\delta_{\perp 12}^r+2\,\mathrm{Re}\beta)}.\tag{5a}$$

The formulas for the transmittance can be written out in a similar form,

$$T_\perp=\frac{\mu_1\,\mathrm{Re}\xi_3}{\mu_3\xi_1}\,\frac{|t_{\perp 12}|^2|t_{\perp 23}|^2e^{-2\,\mathrm{Im}\beta}}{1+R_{\perp 12}R_{\perp 23}e^{-4\,\mathrm{Im}\beta}+R_{\perp 12}^{1/2}R_{\perp 23}^{1/2}e^{-2\,\mathrm{Im}\beta}2\cos(\delta_{\perp 23}^r+\delta_{\perp 12}^r+2\,\mathrm{Re}\beta)},\tag{5b}$$

[4] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, New York, 1941), pp. 492ff.

Parallel Polarization (transverse magnetic, TM)

The corresponding formulas for parallel polarization are

$$r_{\parallel jk}=\frac{\hat{\epsilon}_k\xi_j-\hat{\epsilon}_j\xi_k}{\hat{\epsilon}_k\xi_j+\hat{\epsilon}_j\xi_k}, \qquad t_{\parallel jk}=\frac{H_{\parallel k}{}^0}{H_{\parallel j}{}^0}=\frac{2\hat{\epsilon}_k\xi_j}{\hat{\epsilon}_k\xi_j+\hat{\epsilon}_j\xi_k}, \tag{6}$$

$$r_{\parallel}=\frac{r_{\parallel 12}+r_{\parallel 23}e^{2i\beta}}{1+r_{\parallel 12}r_{\parallel 23}e^{2i\beta}}, \qquad t_{H\parallel}=\frac{t_{\parallel 12}t_{\parallel 23}e^{i\beta}}{1+r_{\parallel 12}r_{\parallel 23}e^{2i\beta}}, \tag{7}$$

$$R_{\parallel}=|r_{\parallel}|^2, \qquad T_{\parallel}=\frac{\mu_3\,\mathrm{Re}(\xi_3/\hat{n}_3{}^2)}{\mu_1\cos\theta_1/n_1}|t_{H\parallel}|^2, \tag{8}$$

$$t_{E\parallel}=\left(\frac{\mu_3\epsilon_1}{\mu_1\hat{\epsilon}_3}\right)^{\frac{1}{2}}t_{H\parallel}=\frac{\mu_3}{\mu_1}\frac{n_1}{\hat{n}_3}t_{H\parallel}, \tag{9}$$

$$\delta_{\parallel}{}^r=\arg r_{\parallel}, \qquad \delta_{\parallel}{}^t=\arg t_{E\parallel}, \tag{10}$$

$$R_{\parallel}=\frac{R_{\parallel 12}+R_{\parallel 23}e^{-4\,\mathrm{Im}\beta}+R_{\parallel 12}{}^{\frac{1}{2}}R_{\parallel 23}{}^{\frac{1}{2}}e^{-2\,\mathrm{Im}\beta}2\cos(\delta_{\parallel 23}{}^r-\delta_{\parallel 12}{}^r+2\,\mathrm{Re}\beta)}{1+R_{\parallel 12}R_{\parallel 23}e^{-4\,\mathrm{Im}\beta}+R_{\parallel 12}{}^{\frac{1}{2}}R_{\parallel 23}{}^{\frac{1}{2}}e^{-2\,\mathrm{Im}\beta}2\cos(\delta_{\parallel 23}{}^r+\delta_{\parallel 12}{}^r+2\,\mathrm{Re}\beta)}, \tag{11a}$$

and for $T_{\parallel}$

$$T_{\parallel}=\frac{\mu_3\,\mathrm{Re}(\xi_3/\hat{n}_3{}^2)}{\mu_1\cos\theta_1/n_1}\,\frac{|t_{\parallel 12}|^2|t_{\parallel 23}|^2e^{-2\,\mathrm{Im}\beta}}{1+R_{\parallel 12}R_{\parallel 23}e^{-4\,\mathrm{Im}\beta}+R_{\parallel 12}{}^{\frac{1}{2}}R_{\parallel 23}{}^{\frac{1}{2}}e^{-2\,\mathrm{Im}\beta}2\cos(\delta_{\parallel 23}{}^r+\delta_{\parallel 12}{}^r+2\,\mathrm{Re}\beta)}. \tag{11b}$$

Equations (5) and (11a) differ from those derived by Vašíček[5,6] for an absorbing film on a metal. Vašíček's equations are not exact because the arguments of the cosine terms in the numerators and denominators of his equations are identical.

According to Poynting's theorem, the real part of the divergence of the complex Poynting vector **S** equals the energy dissipated per unit volume per second from the electromagnetic field at any point.[7] (gaussian c.g.s. units are used in this paper.) It can also be shown that this quantity is related to the conductivity, σ, at the frequency of the field, and the complex electric-field vector **E** by

$$\mathrm{Re}(\nabla\cdot\mathbf{S})=-\tfrac{1}{2}\sigma\mathbf{E}\cdot\mathbf{E}^*=-\sigma\langle E^2\rangle, \tag{12}$$

where $\langle E^2\rangle$ is the mean square of the real electric field. The conductivity is related to the optical constants as follows

$$\mu\sigma=nk\nu, \tag{13}$$

where μ is the magnetic permeability and ν is the frequency of the field. We see from these relationships that the mean-square electric field and the optical properties as a function of frequency define the absorption process at any point of interest.

The formulas for $\langle E^2\rangle$ at any point in the three-phase system of Fig. 1 will now be developed.

[5] A. Vašíček, *Optics of Thin Films* (North-Holland Publishing Company, Amsterdam, 1960), p. 330.
[6] A. Vašíček, Opt. Spectry. (USSR) 11, 128 (1961).
[7] J. A. Stratton, Ref. 4, p. 137.

Field Intensities Perpendicular Polarization

The complex electric field in the incident phase is the vector sum of the incident plane-wave field and that of the reflected plane wave,

$$\mathbf{E}_{\perp 1}=\mathbf{E}_{\perp 1}{}^{0t}\exp(i\mathbf{k}_1{}^t\cdot\mathbf{r}-i\omega t)+\mathbf{E}_{\perp 1}{}^{0r}\exp(i\mathbf{k}_1{}^r\cdot\mathbf{r}-i\omega t). \tag{14}$$

The resultant wave is not plane. Since its x and z components are zero, this field vector has only a y component. At the origin at time zero,

$$E_{\perp 1}=E_{\perp 1}{}^{0t}+E_{\perp 1}{}^{0r}(\mathbf{r},t=0). \tag{15}$$

Now let

$$E_{\perp 1}{}^{0r}=r_{\perp}E_{\perp 1}{}^{0t}=R_{\perp}{}^{\frac{1}{2}}\exp(i\delta_{\perp}{}^r)E_{\perp 1}{}^{0t}, \tag{16}$$

where $r_{\perp}$ is the Fresnel reflection coefficient and $\delta_{\perp}{}^r$ the phase change on reflection for TE polarization. The time-average value of the real part of $E_{\perp 1}$ squared is given by[8]

$$\langle E_{\perp 1}{}^2\rangle=\tfrac{1}{2}(\mathbf{E}_{\perp 1}\cdot\mathbf{E}_{\perp 1}{}^*) \tag{17}$$

and

$$\begin{aligned}\langle E_{\perp 1}{}^2\rangle/\langle E_{\perp 1}{}^{0t2}\rangle &=\{\exp[i(\mathbf{k}_1{}^t\cdot\mathbf{r}-\omega t)]+r_{\perp}\exp[i(\mathbf{k}_1{}^r\cdot\mathbf{r}-\omega t)]\}\\ &\quad\times\{\exp[-i(\mathbf{k}_1{}^t\cdot\mathbf{r}-\omega t)]+r_{\perp}{}^*\exp[-i(\mathbf{k}_1{}^r\cdot\mathbf{r}-\omega t)]\}\end{aligned} \tag{18}$$

or

$$\langle E_{\perp 1}{}^2\rangle=\tfrac{1}{2}(1+R_{\perp})+R_{\perp}{}^{\frac{1}{2}}\cos[\delta_{\perp}{}^r+(\mathbf{k}_1{}^r\cdot\mathbf{r}-\mathbf{k}_1{}^t\cdot\mathbf{r})](E_{\perp 1}{}^{0t})^2. \tag{19}$$

[8] See Ref. 7, p. 136.

All quantities in Eq. (19) are real. Further,

$$(\mathbf{k}_1{}^r\cdot\mathbf{r}-\mathbf{k}_1{}^t\cdot\mathbf{r})=\frac{2\pi}{\lambda}n_1[(x\sin\theta_1-z\cos\theta_1)-(x\sin\theta_1+z\cos\theta_1)] \quad (20)$$
$$=-\frac{4\pi}{\lambda}n_1\cos\theta_1 z=-(4\pi/\lambda)\xi_1 z$$

and

$$\langle E_{\perp1}{}^2\rangle=\tfrac{1}{2}(1+R_\perp)+R_\perp{}^{\frac{1}{2}}\cos[\delta_\perp{}^r-(4\pi/\lambda)\xi_1 z](E_{\perp1}{}^{0t}=1). \quad (21)$$

In phase 3 the electric field has no reflected component. Therefore,

$$\mathbf{E}_{\perp3}=\mathbf{E}_{\perp3}{}^{0t}\exp(i\mathbf{k}_3{}^t\cdot\mathbf{r}'-i\omega t), \quad (22)$$

where $\mathbf{r}'$ refers to boundary 2,3, i.e., $\mathbf{r}'=(x, y, z-h)$; and

$$\langle E_{\perp3}{}^2\rangle=\tfrac{1}{2}(\mathbf{E}_{\perp3}\cdot\mathbf{E}_{\perp3}{}^*)=\tfrac{1}{2}\{E_{\perp3}{}^{0t}E_{\perp3}{}^{0t*}\exp[-2\,\mathrm{Im}(\mathbf{k}_3{}^t\cdot\mathbf{r}')]$$
$$=\tfrac{1}{2}|t_{E\perp}|^2\exp[-(4\pi/\lambda)\,\mathrm{Im}\xi_3(z-h)](E_{\perp1}{}^{0t})^2. \quad (23)$$

In the second phase, the general electric-field vector is given by the sum of two plane waves[9]

$$\mathbf{E}_{\perp2}=\mathbf{E}_{\perp2}{}^{0t}\exp(i\mathbf{k}_2{}^t\cdot\mathbf{r}-i\omega t)+\mathbf{E}_{\perp2}{}^{0r}\exp(i\mathbf{k}_2{}^r\cdot\mathbf{r}-i\omega t). \quad (24)$$

At the origin $E_{\perp2}=E_{\perp1}$ because of the transverse continuity of the electric field, and therefore

$$E_{\perp2}{}^t+E_{\perp2}{}^r=E_{\perp1}{}^t+E_{\perp1}{}^r. \quad (25)$$

The transverse component of the magnetic field is also continuous. Since for a plane wave[10] $\mathbf{H}=(\hat{\epsilon}/\mu)^{\frac{1}{2}}\mathbf{s}\times\mathbf{E}$, for TE polarization,

$$\mathbf{H}_{\mathrm{TE}2}=\left(\frac{\hat{\epsilon}_2}{\mu_2}\right)^{\frac{1}{2}}\mathbf{s}_2{}^t\times\mathbf{E}_{\perp2}{}^t+\left(\frac{\hat{\epsilon}_2}{\mu_2}\right)^{\frac{1}{2}}\mathbf{s}_2{}^r\times\mathbf{E}_{\perp2}{}^r \quad (26)$$

and

$$\mathbf{H}_{\mathrm{TE}2x}=\left(\frac{\hat{\epsilon}_2}{\mu_2}\right)^{\frac{1}{2}}(-\cos\theta_2E_{\perp2}{}^t+\cos\theta_2E_{\perp2}{}^r)=\frac{\xi_2}{\mu_2}(-E_{\perp2}{}^t+E_{\perp2}{}^r). \quad (27)$$

A similar relation holds for phase 1. Therefore, at the boundary

$$\frac{\xi_1\mu_2}{\xi_2\mu_1}(E_{\perp1}{}^t-E_{\perp1}{}^r)=(E_{\perp2}{}^t-E_{\perp2}{}^r)(\mathbf{r}\cdot\hat{z}=0). \quad (28)$$

From Eqs. (25) and (28)

$$\left.\begin{aligned}2E_{\perp2}{}^t&=\left(1+\frac{\xi_1\mu_2}{\xi_2\mu_1}\right)E_{\perp1}{}^t+\left(1-\frac{\xi_1\mu_2}{\xi_2\mu_1}\right)r_\perp E_{\perp1}{}^t\\ 2E_{\perp2}{}^r&=\left(1-\frac{\xi_1\mu_2}{\xi_2\mu_1}\right)E_{\perp1}{}^t+\left(1+\frac{\xi_1\mu_2}{\xi_2\mu_1}\right)r_\perp E_{\perp1}{}^t\end{aligned}\right\}(\mathbf{r}\cdot\hat{z}=0) \quad (29)$$

and

$$E_{\perp2}=E_{\perp2}{}^{0t}\exp(i\mathbf{k}_2{}^t\cdot\mathbf{r}-i\omega t)+E_{\perp2}{}^{0r}\exp(i\mathbf{k}_2{}^r\cdot\mathbf{r}-i\omega t) \quad (30)$$

and

$$\frac{\mathbf{E}_{\perp2}}{E_{\perp1}{}^{0t}}=\exp\left(i\frac{2\pi}{\lambda}n_2\sin\theta_2x-i\omega t\right)\left[(1+r_\perp)\cos\left(\frac{2\pi\xi_2}{\lambda}z\right)+i\frac{\xi_1\mu_2}{\xi_2\mu_1}(1-r_\perp)\sin\left(\frac{2\pi\xi_2}{\lambda}z\right)\right]\hat{y}. \quad (31)$$

Allowing $E_{\perp1}{}^{0t}=1$ and recalling that $\hat{n}_2\sin\theta_2=n_1\sin\theta_1$, which is real, we have

$$E_{\perp2}=\exp\left[i\left(\frac{2\pi}{\lambda}n_1\sin\theta_1x-\omega t\right)\right]\left[(1+r_\perp)\cos\left(\frac{2\pi\xi_2}{\lambda}z\right)+i\frac{\xi_1\mu_2}{\xi_2\mu_1}(1-r_\perp)\sin\left(\frac{2\pi\xi_2}{\lambda}z\right)\right] \quad (32)$$

and

$$\langle E_{\perp2}{}^2\rangle=\tfrac{1}{2}(\mathbf{E}_{\perp2}\cdot\mathbf{E}_{\perp2}{}^*)=\tfrac{1}{2}(E_{\perp2}E_{\perp2}{}^*). \quad (33)$$

Parallel Polarization

In phase 1 the electric field due to incident and reflected plane waves of parallel polarization is

$$\mathbf{E}_{\parallel1}=\mathbf{E}_{\parallel1}{}^{0t}\exp(i\mathbf{k}_1{}^t\cdot\mathbf{r}-i\omega t)+\mathbf{E}_{\parallel1}{}^{0r}\exp(i\mathbf{k}_1{}^r\cdot\mathbf{r}-i\omega t), \quad (34)$$

and the corresponding magnetic field is

$$\mathbf{H}_{\mathrm{TM}1}=\mathbf{H}_{\mathrm{TM}1}{}^t+\mathbf{H}_{\mathrm{TM}1}{}^r=\left(\frac{\epsilon_1}{\mu_1}\right)^{\frac{1}{2}}(\mathbf{s}_1{}^t\times\mathbf{E}_{\parallel1}{}^t+\mathbf{s}_1{}^r\times\mathbf{E}_{\parallel1}{}^r), \quad (35)$$

where $\mathbf{H}_{\mathrm{TM}1}$ refers to the magnetic field in phase 1 for TM polarization. The phase relationship between initial and reflected waves is somewhat arbitrary. The usual conventions result if we set

$$\mathbf{H}_{\mathrm{TM}1}{}^r=r_\parallel\mathbf{H}_{\mathrm{TM}1}{}^t=R_\parallel{}^{\frac{1}{2}}\exp(i\delta_\parallel{}^r)\mathbf{H}_{\mathrm{TM}1}{}^t(\mathbf{r}\cdot\hat{z}=0), \quad (36)$$

where $r_\parallel$ is the Fresnel reflection coefficient for TM polarization and $\delta_\parallel{}^r$ is the phase change on reflection. From this and the fact that $\mathbf{E}=-(\mu/\hat{\epsilon})^{\frac{1}{2}}\mathbf{s}\times\mathbf{H}$,[10]

$$\mathbf{E}_{\parallel1}{}^r=-\left(\frac{\mu_1}{\epsilon_1}\right)^{\frac{1}{2}}\mathbf{s}_1{}^r\times\mathbf{H}_{\mathrm{TM}1}{}^r \quad (37)$$

and

$$\mathbf{E}_{\parallel1x}{}^r=-\cos\theta_1r_\parallel E_{\parallel1}{}^t$$

and

$$\mathbf{E}_{\parallel1z}{}^r=\sin\theta_1r_\parallel E_{\parallel1}{}^t(\mathbf{r}\cdot\hat{z}=0) \quad (38)$$

and, setting the signs of $\mathbf{E}_{\parallel1}{}^r$ and $\mathbf{E}_{\parallel1}{}^t$ the same as $\mathbf{H}_{\mathrm{TM}1}{}^r$ and $\mathbf{H}_{\mathrm{TM}1}{}^t$

$$E_{\parallel1}{}^r=r_\parallel E_{\parallel1}{}^t(\mathbf{r}\cdot\hat{z}=0). \quad (39)$$

[9] See Ref. 7, p. 511.
[10] M. Born and E. Wolf, Ref. 3, p. 23.

Now we can write down the time average of the square of the real electric field, viz.,

$$\langle E_{\parallel 1}^2\rangle=\tfrac{1}{2}(\mathbf{E}_{\parallel 1}\cdot\mathbf{E}_{\parallel 1}^*)=\langle E_{\parallel 1x}^2\rangle+\langle E_{\parallel 1z}^2\rangle, \quad (40)$$

where

$$\langle E_{\parallel 1x}^2\rangle=\cos^2\theta_1\left[\tfrac{1}{2}(1+R_\parallel)-R_\parallel^{\frac{1}{2}}\cos(\delta_\parallel{}^r-4\pi(z/\lambda)\xi_1)\right](E_{\parallel 1}{}^{0t})^2 \quad (41)$$

and

$$\langle E_{\parallel 1z}^2\rangle=\sin^2\theta_1\left[\tfrac{1}{2}(1+R_\parallel)+R_\parallel^{\frac{1}{2}}\times\cos\left(\delta_\parallel{}^r-4\pi\frac{z}{\lambda}\xi_1\right)\right](E_{\parallel 1}{}^{0t})^2. \quad (42)$$

In the second phase

$$\mathbf{E}_{\parallel 2}=\mathbf{E}_{\parallel 2}{}^{0t}\exp\eta^t+\mathbf{E}_{\parallel 2}{}^{0r}\exp\eta^r, \quad (43)$$

where $\eta^t=(i\mathbf{k}_2{}^t\cdot\mathbf{r}-i\omega t)$ and $\eta^r=(i\mathbf{k}_2{}^r\cdot\mathbf{r}-i\omega t)$. This gives

$$\mathbf{E}_{\parallel 2}=-\left(\frac{\mu_2}{\hat{\epsilon}_2}\right)^{\frac{1}{2}}(\mathbf{s}_2{}^t\times\mathbf{H}_{\mathrm{TM}2}{}^{0t}\exp\eta^t+\mathbf{s}_2{}^r\times\mathbf{H}_{\mathrm{TM}2}{}^{0r}\exp\eta^r). \quad (44)$$

By examining Maxwell's equations (given for example on p. 609 of Ref. 3) for the case of no free-charge density and with conductivity, σ, formally accounted for by a complex dielectric constant, we can see that they are unchanged by the substitution of $\mathbf{H}$ for $\mathbf{E}$ and $-\mu$ for $\hat{\epsilon}$. Therefore, this same substitution can be made in equations derived from Maxwell's equations, i.e., the roles of $\mathbf{E}$ and $\mathbf{H}$ and $\hat{\epsilon}$ and $-\mu$ can be interchanged. Note also that since $\hat{n}=(\mu\hat{\epsilon})^{\frac{1}{2}}$ this interchange doesn't change $\hat{n}$ or ξ. However, $r_\perp$ is changed to $r_\parallel$ and *vice versa.* Making these substitutions in Eqs. (29) we get

$$\left.\begin{aligned}\mathbf{H}_{\mathrm{TM}2}{}^t&=\frac{1}{2}\left[\left(1+\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)+\left(1-\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)r_\parallel\right]\mathbf{H}_{\mathrm{TM}1}{}^t\\ &\text{and}\\ \mathbf{H}_{\mathrm{TM}2}{}^r&=\frac{1}{2}\left[\left(1-\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)+\left(1+\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)r_\parallel\right]\mathbf{H}_{\mathrm{TM}1}{}^t\end{aligned}\right\}(\mathbf{r}\cdot\hat{z}=0). \quad (45)$$

Recalling that

$$\mathbf{H}_{\mathrm{TM}1}{}^t=\left(\frac{\epsilon_1}{\mu_1}\right)^{\frac{1}{2}}\mathbf{s}_1{}^t\times\mathbf{E}_{\parallel 1}{}^t \quad (46)$$

and using Eqs. (37) and (44) we have

$$\left.\begin{aligned}\mathbf{E}_{\parallel 2}=&-\left(\frac{\mu_2}{\hat{\epsilon}_2}\right)^{\frac{1}{2}}\left\{\mathbf{s}_2{}^t\times\left[\frac{1}{2}\left(1+\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)+\left(1-\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)r_\parallel\right]\left(\frac{\epsilon_1}{\mu_1}\right)^{\frac{1}{2}}\mathbf{s}_1{}^t\times\mathbf{E}_{\parallel 1}{}^{0t}\exp\eta^t\right.\\ &\left.+\mathbf{s}_2{}^r\times\left[\frac{1}{2}\left(1-\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)+\left(1+\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)r_\parallel\right]\left(\frac{\epsilon_1}{\mu_1}\right)^{\frac{1}{2}}\mathbf{s}_1{}^t\times\mathbf{E}_{\parallel 1}{}^{0t}\exp\eta^r\right\}\\ =&(\cos\theta_2,0,-\sin\theta_2)E_{\parallel 1}{}^{0t}\left(\frac{\mu_2\epsilon_1}{\mu_1\hat{\epsilon}_2}\right)^{\frac{1}{2}}\frac{1}{2}\left\{\left[\left(1+\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)+\left(1-\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)r_\parallel\right]\exp\eta^t\right\}\\ &+(-\cos\theta_2,0,-\sin\theta_2)E_{\parallel 1}{}^{0t}\left(\frac{\mu_2\epsilon_1}{\mu_1\hat{\epsilon}_2}\right)^{\frac{1}{2}}\frac{1}{2}\left\{\left[\left(1-\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)+\left(1+\frac{\xi_1\hat{\epsilon}_2}{\xi_2\epsilon_1}\right)r_\parallel\right]\exp\eta^r\right\}\end{aligned}\right\} \quad (47)$$

or

$$\left.\begin{aligned}E_{\parallel 2x}=&\exp\left(i\frac{2\pi}{\lambda}\hat{n}_2\sin\theta_2x-i\omega t\right)\left[\cos\theta_1(1-r_\parallel)\cos\left(\frac{2\pi}{\lambda}\xi_2z\right)+i\frac{\mu_2\xi_2}{\mu_1\hat{n}_2^2}n_1(1+r_\parallel)\sin\left(\frac{2\pi}{\lambda}\xi_2z\right)\right]E_{\parallel 1}{}^{0t}\\ &\text{and}\\ E_{\parallel 2z}=&-\exp\left(i\frac{2\pi}{\lambda}\hat{n}_2\sin\theta_2x-i\omega t\right)\left[\frac{\mu_2}{\mu_1}\frac{\hat{n}_2\sin\theta_2}{\hat{n}_2^2}n_1(1+r_\parallel)\cos\left(\frac{2\pi}{\lambda}\xi_2z\right)+\frac{\hat{n}_2\sin\theta_2}{\xi_2}\cos\theta_1(1-r_\parallel)\sin\left(\frac{2\pi}{\lambda}\xi_2z\right)\right]E_{\parallel 1}{}^{0t}\end{aligned}\right\}. \quad (48)$$

Keep in mind that $\hat{n}_2\sin\theta_2=n_1\sin\theta_1$ which is real. Now

$$\langle E_{\parallel 2}^2\rangle=\tfrac{1}{2}(\mathbf{E}_{\parallel 2}\cdot\mathbf{E}_{\parallel 2}^*)=\tfrac{1}{2}(|\mathbf{E}_{\parallel 2x}|^2+|\mathbf{E}_{\parallel 2z}|^2)=\langle E_{\parallel 2x}^2\rangle+\langle E_{\parallel 2z}^2\rangle, \quad (49)$$

since the y component is zero.

In the third phase

$$\mathbf{H}_{\mathrm{TM}3}{}^t=\mathbf{H}_{\mathrm{TM}1}{}^t t_{H\parallel}=\left(\frac{\epsilon_1}{\mu_1}\right)^{\frac{1}{2}}\mathbf{s}_1{}^t\times\mathbf{E}_{\parallel 1}{}^t t_{H\parallel}(\mathbf{r}\cdot\hat{z}=h), \quad (50)$$

where $t_{H\parallel}$ is given by Eq. (7). Therefore,

$$\mathbf{E}_{\parallel 3}{}^t=-\left(\frac{\mu_3}{\hat{\epsilon}_3}\right)^{\frac{1}{2}}\mathbf{s}_3{}^t\times\mathbf{H}_{\mathrm{TM}3}{}^t=(\cos\theta_3,0,-\sin\theta_3)\times\left(\frac{\mu_3\epsilon_1}{\hat{\epsilon}_3\mu_1}\right)^{\frac{1}{2}}t_{H\parallel}E_{\parallel 1}{}^t(\mathbf{r}\cdot\hat{z}=h) \quad (51)$$

and

$$E_{\parallel 3}{}^{t}=\left(\frac{\mu_3\epsilon_1}{\hat{\epsilon}_3\mu_1}\right)^{\frac{1}{2}} t_{H\parallel}E_{\parallel 1}{}^{t}=t_{E\parallel}E_{\parallel 1}{}^{t}(\mathbf{r}\cdot\hat{z}=h). \tag{52}$$

The general expression for the electric field anywhere in the third phase is given by

$$\mathbf{E}_{\parallel 3}=\mathbf{E}_{\parallel 3}{}^{0t}\exp(i\mathbf{k}_3{}^{t}\cdot\mathbf{r}'-i\omega t), \tag{53}$$

where $\mathbf{r}'$ is the position vector relative to the point $(0,0,h)$, which is on the 2,3 boundary. Therefore, from Eqs. (51)–(53)

$$\begin{aligned}\mathbf{E}_{\parallel 3x}&=\cos\theta_3 t_{E\parallel}E_{\parallel 1}{}^{0t}\exp\left(i\frac{2\pi}{\lambda}\times\hat{n}_3\sin\theta_3-i\omega t\right)\\&\quad\times\exp\left[i\frac{2\pi}{\lambda}\xi_3(z-h)\right]\\ \mathbf{E}_{\parallel 3z}&=-\sin\theta_3 t_{E\parallel}E_{\parallel 1}{}^{0t}\exp\left(i\frac{2\pi}{\lambda}\times\hat{n}_3\sin\theta_3-i\omega t\right)\\&\quad\times\exp\left[i\frac{2\pi}{\lambda}\xi_3(z-h)\right]\end{aligned} \tag{54}$$

and

$$\langle E_{\parallel 3}{}^2\rangle=\tfrac{1}{2}|t_{E\parallel}|^2\exp[-4\pi\,\mathrm{Im}\xi_3(z-h)/\lambda]E_{\parallel 1}{}^{0t2} \tag{55}$$

$$\langle E_{\parallel 3x}{}^2\rangle=\frac{1}{2}\left|\frac{\xi_3}{\hat{n}_3}t_{E\parallel}\right|^2\exp[-4\pi\,\mathrm{Im}\xi_3(z-h)/\lambda]E_{\parallel 1}{}^{0t2} \tag{56}$$

$$\langle E_{\parallel 3z}{}^2\rangle=\frac{1}{2}\left|\frac{n_1\sin\theta_1}{\hat{n}_3}t_{E\parallel}\right|^2\exp[-4\pi\,\mathrm{Im}\xi_3(z-h)/\lambda]E_{\parallel 1}{}^{0t2}. \tag{57}$$

B. Two-Phase System

General Equations

The general equations above encompass the important simpler two-phase case. Some of the equations for this case will be stated explicitly and important aspects will be described. Some aspects of the two-phase case and for the case of a very thin film between two phases have been discussed by Harrick[1] and by Harrick and du Pré.[11] They give equations for the complex amplitudes of the electric fields for the nonabsorbing, nonmagnetic, isotropic case, at angles greater than critical.

The reflectance and transmittance for perpendicular polarization follow directly from the above three-phase equations and are given by

$$R_\perp=|r_\perp|^2,\quad\text{and}\quad T_\perp=\frac{\mu_1}{\mu_2}\frac{\mathrm{Re}\xi_2}{\xi_1}|t_{E\perp}|^2, \tag{58}$$

where

$$r_\perp=r_{\perp 12}=\frac{\mu_2\xi_1-\mu_1\xi_2}{\mu_2\xi_1+\mu_1\xi_2},\quad\text{and}\quad t_{E\perp}=t_{\perp 12}=\frac{2\mu_2\xi_1}{\mu_2\xi_1+\mu_1\xi_2}. \tag{59}$$

[11] N. J. Harrick and F. K. du Pré, Appl. Opt. 5, 1739 (1966).

Likewise for parallel polarization we have

$$R_\parallel=|r_\parallel|^2,\quad\text{and}\quad T_\parallel=\frac{\mu_2}{\mu_1}\frac{n_1{}^2}{\xi_1}\mathrm{Re}\frac{\xi_2}{\hat{n}_2{}^2}|t_{H\parallel}|^2, \tag{60}$$

where

$$r_\parallel=r_{\parallel 12}=\frac{\hat{\epsilon}_2\xi_1-\epsilon_1\xi_2}{\hat{\epsilon}_2\xi_1+\epsilon_1\xi_2}\quad t_{H\parallel}=t_{\parallel 12}=\frac{2\hat{\epsilon}_2\xi_1}{\hat{\epsilon}_2\xi_1+\epsilon_1\xi_2}, \tag{61}$$

and

$$t_{E\parallel}=\frac{\mu_2}{\mu_1}\frac{n_1}{\hat{n}_2}t_{H\parallel}. \tag{62}$$

The mean-square electric fields are given by

$$\langle E_{\perp 1}{}^2\rangle=[\tfrac{1}{2}(1+R_\perp)+R_\perp{}^{\frac{1}{2}}\cos(\delta_\perp{}^r-4\pi(z/\lambda)\xi_1)]E_{\perp 1}{}^{0t2}, \tag{63}$$

where $r_\perp=R_\perp{}^{\frac{1}{2}}\exp(i\delta_\perp{}^r)$ and z is negative in phase 1,

$$\langle E_{\perp 2}{}^2\rangle=\tfrac{1}{2}|t_{E\perp}|^2\exp(-4\pi(z/\lambda)\,\mathrm{Im}\xi_2)E_{\perp 1}{}^{0t2} \tag{63a}$$

$$\langle E_{\parallel 2}{}^2\rangle=\tfrac{1}{2}|t_{E\parallel}|^2\exp(-4\pi\,\mathrm{Im}\xi_2(z/\lambda))E_{\parallel 1}{}^{0t2} \tag{64}$$

$$\langle E_{\parallel 1x}{}^2\rangle=\cos^2\theta_1[\tfrac{1}{2}(1+R_\parallel)-R_\parallel{}^{\frac{1}{2}}\cos(\delta_\parallel{}^r-4\pi(z/\lambda)\xi_1)]E_{\parallel 1}{}^{0t2} \tag{65}$$

$$\langle E_{\parallel 1z}{}^2\rangle=\sin^2\theta_1[\tfrac{1}{2}(1+R_\parallel)+R_\parallel{}^{\frac{1}{2}}\cos(\delta_\parallel{}^r-4\pi(z/\lambda)\xi_1)]E_{\parallel 1}{}^{0t2}, \tag{66}$$

where $r_\parallel=R_\parallel{}^{\frac{1}{2}}\exp(i\delta_\parallel{}^r)$,

$$\langle E_{\parallel 2x}{}^2\rangle=\frac{1}{2}\left|\frac{\xi_2}{\hat{n}_2}t_{E\parallel}\right|^2\exp\left(-4\pi\frac{z}{\lambda}\mathrm{Im}\xi_2\right)E_{\parallel 1}{}^{0t2} \tag{67}$$

$$\langle E_{\parallel 2z}{}^2\rangle=\frac{1}{2}\left|\frac{n_1\sin\theta_1}{\hat{n}_2}t_{E\parallel}\right|^2\exp\left(-4\pi\frac{z}{\lambda}\mathrm{Im}\xi_2\right)E_{\parallel 1}{}^{0t2}. \tag{68}$$

It is convenient to assume $E_{\perp 1}{}^{0t}$ and $E_{\parallel 1}{}^{0t}$ to be unity and to leave them out of the above equations. In this case $\langle E_{\perp 1}{}^{t2}\rangle$ and $\langle E_{\parallel 1}{}^{t2}\rangle$ each equal $\frac{1}{2}$.

Examples of Fields in a Two-Phase System

For two transparent phases, at the boundary

$$\langle E_{\perp 1}{}^2\rangle=\langle E_{\perp 2}{}^2\rangle=\frac{1}{2}\left|\frac{2\mu_2\xi_1}{\mu_2\xi_1+\mu_1\xi_2}\right|^2(E_{\perp 1}{}^{0t})^2(\mathbf{r}\cdot\hat{z}=0) \tag{69}$$

$$\langle E_{\parallel 2}{}^2\rangle=\frac{1}{2}\left|\left(\frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}\right)^{\frac{1}{2}}\frac{2\epsilon_2\xi_1}{\epsilon_2\xi_1+\epsilon_1\xi_2}\right|^2(E_{\parallel 1}{}^{0t})^2(\mathbf{r}\cdot\hat{z}=0). \tag{70}$$

If we make the usual assumption that $\mu=1$, and the convenient assumption that $E_{\perp 1}{}^{0t}$ and $E_{\parallel 1}{}^{0t}$ equal unity

$$\left.\begin{aligned}\langle E_\perp{}^2\rangle&=\frac{2\xi_1{}^2}{(\xi_1+\xi_2)^2}\quad\text{for }\theta_1\le\theta_c\\&=\frac{2\xi_1{}^2}{n_1{}^2-n_2{}^2}\quad\text{for }\theta_1\ge\theta_c\end{aligned}\right\}\quad\begin{aligned}\mathbf{r}\cdot\hat{z}&=0\\ \mu&=1\\ E_{\perp 1}{}^{0t}&=1\end{aligned} \tag{71}$$

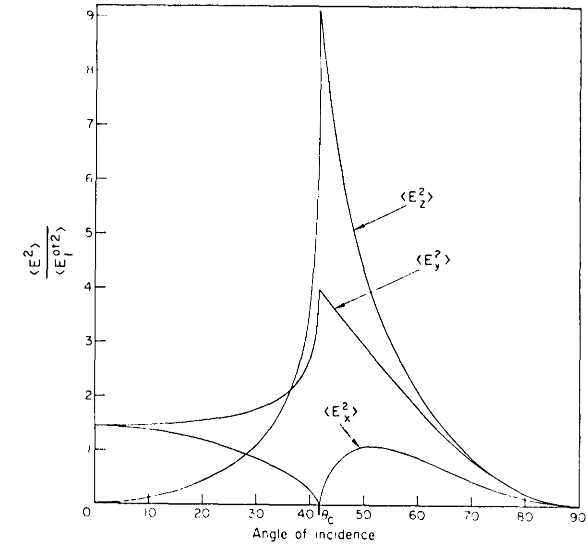

FIG. 2. Mean-square electric field in air at the surface of glass as a function of internal angle of incidence for $\perp$ polarization, $\langle E_y^2\rangle$, and for $\|$ polarization, $\langle E_x^2\rangle+\langle E_z^2\rangle$. The indices are: $n_1=1.51$, $n_2=1.0$, and $k_2=0$.

$$\left.\begin{aligned}\langle E_{\|2}^2\rangle &= 2\left(\frac{\xi_1}{n_2\cos\theta_1+n_1\cos\theta_2}\right)^2 \quad \text{for } \theta_1\le\theta_c \\ &= 2\frac{\xi_1^2}{(n_2\cos\theta_1)^2-(n_1\cos\theta_2)^2} \quad \text{for } \theta_1\ge\theta_c\end{aligned}\right\}. \quad \begin{aligned}\mathbf{r}\cdot\hat{\mathbf{z}}&=0\\ \mu&=1\\ E_{\|1}^{0t}&=1\end{aligned} \tag{72}$$

At the critical angle, the TE and TM fields at the interface in phase 2 are a maximum given by

$$\left.\begin{aligned}\langle E_\perp^2\rangle &= 2\\ \langle E_{\|x}^2\rangle &= 0\\ \langle E_{\|z}^2\rangle &= 2(n_1/n_2)^2\end{aligned}\right\}. \quad \begin{aligned}E_\perp^{0t}&=1\\ \mathbf{r}\cdot\hat{\mathbf{z}}&=0\\ \theta&=\theta_c\end{aligned} \tag{73}$$

At grazing incidence all fields go to zero at the boundary because the incident and totally reflected field are exactly out of phase. This is also true if phase 2 is absorbing. At normal incidence

$$\left.\begin{aligned}\langle E_\perp^2\rangle=\langle E_{\|x}^2\rangle &= 2\left(\frac{n_1}{n_1+n_2}\right)^2\\ \langle E_{\|z}^2\rangle &= 0\end{aligned}\right\}. \quad \begin{aligned}E_\perp^{0t}&=1\\ \mathbf{r}\cdot\hat{\mathbf{z}}&=0\\ \theta_1&=0\end{aligned} \tag{74}$$

In Fig. 2 are plotted the fields at the surface of glass at which internal reflection is occurring. Note the simplifying features discussed above. In Fig. 3, curves for a transparent silicon prism are given. Note the extremely high field perpendicular to the surface near the critical angle. In the case of germanium ($n_1=4$) the value of $\langle E_{\|2}^2\rangle/\langle E_{\|1}^{t2}\rangle$ would reach 64 at the critical angle. The maximum value of $\langle E_{\perp 2}^2\rangle/\langle E_{\perp 1}^{t2}\rangle$ is always 4.

Figure 4 shows what happens to the fields at the boundary in a phase of $n_2=1.4$ against germanium in the infrared. The angle of incidence is the critical angle where the z field is very large compared to the field in the first phase. As the second phase becomes absorbing the z field decreases but is still large when $\kappa_2=0.05$, a typical value for a strong absorption band of an organic in the infrared ($\kappa_2=k_2/n_2$, cf. Appendix B). At the same time, the x component increases from zero. As κ_2 gets very large, typical of a metal, the z component gets small while the x and y components remain rather constant.

From Eqs. (21) and (40), more details of reflection from absorbing phases become evident. One interesting point is that while many authors state or imply that there is a node at the surface of a highly conductive metal, this is not true for parallel polarization. For example, at $\theta_1=45°$ and $R_\|$ near unity (typical of silver, gold, aluminum, etc.), Eqs. (40)–(42) show that the total parallel field, $\langle E_x^2\rangle+\langle E_z^2\rangle$, is constant and equal to about twice the incident field, right up to the boundary, regardless of how great k_2 becomes. Other angles, except near normal and near grazing also give large fields at the surface in the case of parallel polarization. This has important implications in obtaining spectra of molecules chemisorbed on metal surfaces. Figure 5 shows the fields for the case of reflection, where $n_2/n_1=0.2$ and $\kappa_2=10$. As is always the case at 45°, the total parallel field in phase 1 is constant, right up to the boundary.

C. N-Phase System

The methods of calculation of this section are based upon the work of Abelès[12] for a stratified medium. Insofar as practical, the symbolism has been made parallel

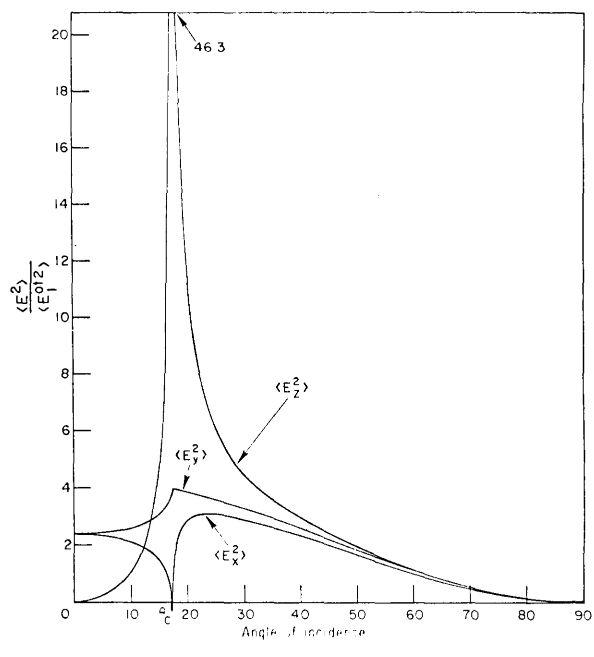

FIG. 3. Mean-square electric field in air at the surface of silicon as a function of internal angle of incidence. The indices are: $n_1=3.4$, $n_2=1.0$, $k_2=0$.

to that of Ref. 3, Sec. 1.6. However, the formulas for reflection coefficients, etc., which are similar in form to those of Ref. 3, are more general, applying to absorbing media. Their component terms must be evaluated as described in this paper. Assume homogeneous, isotropic plane-bounded layers with otherwise arbitrary optical parameters, between semi-infinite initial and final phases. Light is incident from the initial phase, which is conveniently assumed transparent. We first calculate the reflectance, transmittance, and phase changes. We then derive expressions for the mean-square fields everywhere, in terms of the reflectance (or transmittance) and phase changes.

Reflectance, Transmittance, and Phase Changes

In this general case, with $N-1$ surfaces of discontinuity at $z=z_k(k=1, 2, \cdots N-1)$, there is a matrix M_k characteristic of each space $z_{k-1}\leq z\leq z_k$ and for the final region $z\geq z_{N-1}$. The tangential fields at the first boundary, $z=z_1=0$, are related to those at the final boundary, $z=z_{N-1}$, by

$$\begin{bmatrix} U_1 \\ V_1 \end{bmatrix} = M_2 M_3 \cdots M_{N-1} \begin{bmatrix} U_{N-1} \\ V_{N-1} \end{bmatrix} = M \begin{bmatrix} U_{N-1} \\ V_{N-1} \end{bmatrix}, \quad (75)$$

where M is the characteristic matrix of the plane bounded layers as a group and U_k and V_k are the tangential components of the field amplitudes at the boundary k. For TE polarization, $U_1=E_y{}^0$ and $V_1=H_x{}^0$ at boundary 1, and the analogous relation holds for boundary 2. For TM polarization at boundary k,

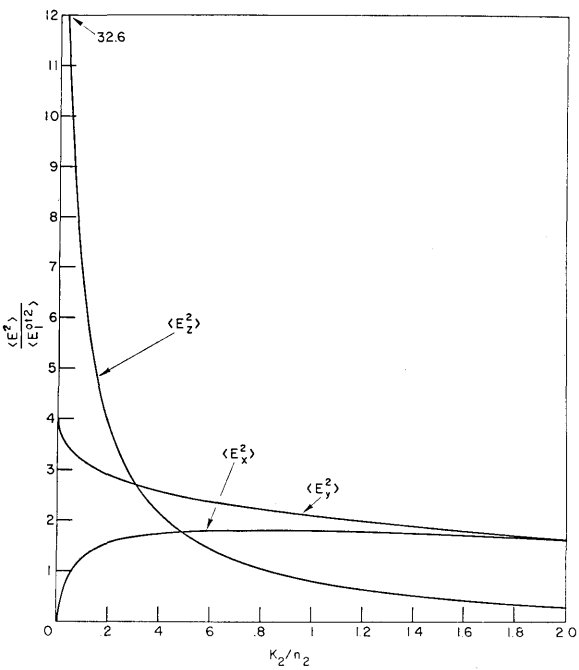

FIG. 4. Dependence of mean-square fields in the last medium on the extinction coefficient. The optical parameters are: $n_1=4$, $n_2=1.4$, k_2 varies, and $\theta_1=20.5°$.

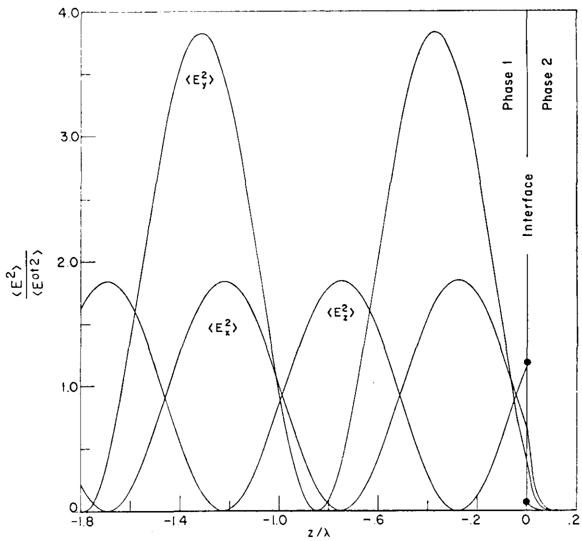

FIG. 5. Standing-wave fields in phase 1 for metallic reflection. The optical parameters are: $n_1=1.5$, $n_2=0.3$, $k_2=3$, and $\theta_1=45°$

$U_k=H_y{}^0$ and $V_k=E_x{}^0$. The characteristic matrix for the jth layer is given for TE polarization by

$$M_j=\begin{bmatrix} \cos\beta_j & \dfrac{-i}{p_j}\sin\beta_j \\ -ip_j\sin\beta_j & \cos\beta_j \end{bmatrix}, \quad \text{(TE polarization)} \quad (76)$$

where $\beta_j=(2\pi/\lambda)\xi_j h_j$, and $p_j=(\hat{\epsilon}_j/\mu_j)^{\frac{1}{2}}\cos\theta_j$. For TM polarization we can interchange $\hat{\epsilon}$ and $-\mu$ to get

$$M_j=\begin{bmatrix} \cos\beta_j & \dfrac{-i}{q_j}\sin\beta_j \\ -iq_j\sin\beta_j & \cos\beta_j \end{bmatrix}, \quad \text{(TM polarization)} \quad (77)$$

where $q_j=(\mu_j/\hat{\epsilon}_j)^{\frac{1}{2}}\cos\theta_j$. From (75) we can derive the reflection and transmission coefficients of the stack

$$r_\perp=\frac{E_{y1}{}^{0r}}{E_{y1}{}^{0t}}=\frac{(m_{11}+m_{12}p_N)p_1-(m_{21}+m_{22}p_N)}{(m_{11}+m_{12}p_N)p_1+(m_{21}+m_{22}p_N)} \quad (78)$$

$$t_{E\perp}=\frac{E_{yN}{}^{0t}}{E_{y1}{}^{0t}}=\frac{2p_1}{(m_{11}+m_{12}p_N)p_1+(m_{21}+m_{22}p_N)} \quad (79)$$

$$r_{\parallel}=\frac{H_{y1}{}^{0r}}{H_{y1}{}^{0t}}=\frac{(m_{11}+m_{12}q_N)q_1-(m_{21}+m_{22}q_N)}{(m_{11}+m_{12}q_N)q_1+(m_{21}+m_{22}q_N)} \quad (80)$$

$$t_{H\parallel}=\frac{H_{yN}{}^{0t}}{H_{y1}{}^{0t}}=\frac{2q_1}{(m_{11}+m_{12}q_N)q_1+(m_{21}+m_{22}q_N)} \quad (81)$$

$$t_{E\parallel}=(\mu_N/\mu_1)(n_1/\hat{n}_N)t_{H\parallel}. \quad (82)$$

In these equations, the m_{ij} are the elements of the matrix M given by (75), which is characteristic of the stratified medium.

From the above equations

$$R_{\perp}=|r_{\perp}|^2,\quad \delta_{\perp}^r=\arg r_{\perp} \tag{83}$$

$$T_{\perp}=\frac{\mu_1 \operatorname{Re}(\hat{n}_N \cos\theta_N)}{\mu_N n_1 \cos\theta_1}|t_{E\perp}|^2,\quad \delta_{\perp}^t=\arg t_{E\perp} \tag{84}$$

$$R_{\parallel}=|r_{\parallel}|^2,\quad \delta_{\parallel}^r=\arg r_{\parallel} \tag{85}$$

$$T_{\parallel}=\frac{\mu_N \operatorname{Re}(\hat{n}_N \cos\theta_N/\hat{n}_N^2)}{\mu_1 n_1 \cos\theta_1/n_1^2}|t_{H\parallel}|^2,\quad \delta_{\parallel}^t=\arg t_{E\parallel}. \tag{86}$$

Field Intensities

The fields at a general point in the stratified medium will now be derived. It is convenient to use the reciprocal of M, which we will define as N, i.e., $NM=I$ where I is the unit matrix. From Ref. 12 we know that $Q_1=M_2(z)Q_2(z)$, where Q_1 is a matrix at boundary 1, $M_2(z)$ is the characteristic matrix of phase 2 evaluated at a position z in that phase, and $Q_2(z)$ is the matrix relating to the fields at position z in phase 2. The matrix Q is defined by Eqs. (90) and (91) below. Multiplying both sides from the left by $N_2(z)$, the reciprocal of $M_2(z)$, we get

$$Q_2(z)=N_2(z)Q_1, \tag{87}$$

and in general

$$Q_k(z)=N_k(z)\prod_{j=k-1}^{2} N_jQ_1, \tag{88}$$

or

$$Q_k(z)=N_k(z)\prod_{j=k}^{N-1} M_jQ_{N-1}. \tag{89}$$

So Eqs. (88) and (89) give the fields throughout the stratified medium in terms of the fields at the first or last boundary, which are in turn given by r and t.

To be more explicit, for TE polarization,

$$Q_1=\begin{bmatrix}U_1\\V_1\end{bmatrix}=\begin{bmatrix}E_{y1}^0\\H_{x1}^0\end{bmatrix}=\begin{bmatrix}E_{y1}^{0t}+E_{y1}^{0r}\\H_{x1}^{0t}+H_{x1}^{0r}\end{bmatrix}=\begin{bmatrix}1+r_{\perp}\\-p_1(1-r_{\perp})\end{bmatrix}E_{y1}^{0t} \tag{90}$$

$$Q_{N-1}=\begin{bmatrix}U_{N-1}\\V_{N-1}\end{bmatrix}=\begin{bmatrix}E_{y(N-1)}^0\\H_{x(N-1)}^0\end{bmatrix}=\begin{bmatrix}E_{y(N-1)}^{0t}\\H_{x(N-1)}^{0t}\end{bmatrix}=\begin{bmatrix}t_{E\perp}\\p_Nt_{E\perp}\end{bmatrix}E_{y1}^{0t} \tag{91}$$

$$N_k(z)=\begin{bmatrix}\cos\left(\frac{2\pi}{\lambda}\xi_k(z-z_{k-1})\right) & \frac{i}{p_k}\sin\left(\frac{2\pi}{\lambda}\xi_k(z-z_{k-1})\right)\\ ip_k\sin\left(\frac{2\pi}{\lambda}\xi_k(z-z_{k-1})\right) & \cos\left(\frac{2\pi}{\lambda}\xi_k(z-z_{k-1})\right)\end{bmatrix}. \tag{92}$$

The values of the M_j are given by Eq. (77), or N_j can be had by merely changing the signs of the negative terms inside the brackets. From Eqs. (88) or (89) we can now calculate

$$Q_k(z)\equiv\begin{bmatrix}U_k(z)\\V_k(z)\end{bmatrix} \tag{93}$$

and

$$E_{1k}=U_k(z)\exp\left(i\frac{2\pi}{\lambda}n_1\sin\theta_1x-i\omega t\right). \tag{94}$$

This equation can be compared directly with Eq. (32). The mean-square field we seek is given by

$$\langle E_{1k}^2\rangle=\tfrac{1}{2}E_{1k}E_{1k}^*=\tfrac{1}{2}|U_k(z)|^2. \tag{95}$$

For the magnetic field we have

$$H_{\mathrm{TE}kx}=V_k(z)\exp\left(i\frac{2\pi}{\lambda}n_1\sin\theta_1x-i\omega t\right) \tag{96}$$

and

$$H_{\mathrm{TE}kz}=W_k(z)\exp\left(i\frac{2\pi}{\lambda}n_1\sin\theta_1x-i\omega t\right) \tag{97}$$

giving

$$\mathbf{H}_{\mathrm{TE}k}=(V_k(z),0,W_k(z))\exp\left(i\frac{2\pi}{\lambda}n_1\sin\theta_1x-i\omega t\right). \tag{98}$$

Here

$$W_k(z)=n_1\sin\theta_1U_k(z)/\mu_k. \tag{99}$$

We can obtain the formulas for TM polarization immediately by interchanging H and E, and $\hat{\epsilon}$ and $-\mu$ in the formulas for TE polarization. Thus,

$$Q_1=\begin{bmatrix}U_1\\V_1\end{bmatrix}=\begin{bmatrix}H_{y1}^0\\E_{x1}^0\end{bmatrix}=\begin{bmatrix}H_{y1}^{0t}+H_{y1}^{0r}\\E_{x1}^{0t}+E_{x1}^{0r}\end{bmatrix}=\begin{bmatrix}1+r_{\parallel}\\-q_1(1-r_{\parallel})\end{bmatrix}H_{y1}^{0t} \tag{100}$$

$$Q_{N-1}=\begin{bmatrix}U_{N-1}\\V_{N-1}\end{bmatrix}=\begin{bmatrix}H_{y(N-1)}^0\\E_{x(N-1)}^0\end{bmatrix}=\begin{bmatrix}H_{y(N-1)}^{0t}\\E_{x(N-1)}^{0t}\end{bmatrix}=\begin{bmatrix}t_{H\parallel}\\-q_Nt_{H\parallel}\end{bmatrix}H_{y1}^{0t} \tag{101}$$

[12] F. Abelès, Ann. Phys. (Paris) 5, 596 (1950).

$$N_k(z)=\begin{bmatrix} \cos\left(\frac{2\pi}{\lambda}\xi_k(z-z_{k-1})\right) & \frac{i}{q_k}\sin\left(\frac{2\pi}{\lambda}\xi_k(z-z_{k-1})\right) \\ iq_k\sin\left(\frac{2\pi}{\lambda}\xi_k(z-z_{k-1})\right) & \cos\left(\frac{2\pi}{\lambda}\xi_k(z-z_{k-1})\right) \end{bmatrix} \quad (102)$$

$$H_{\mathrm{TM}k}=U_k(z)\exp\left(i\frac{2\pi}{\lambda}n_1\sin\theta_1 x-i\omega t\right) \quad (103)$$

$$E_{\parallel kx}=V_k(z)\exp\left(i\frac{2\pi}{\lambda}n_1\sin\theta_1 x-i\omega t\right) \quad (104)$$

$$E_{\parallel kz}=W_k(z)\exp\left(i\frac{2\pi}{\lambda}n_1\sin\theta_1 x-i\omega t\right), \quad (105)$$

giving

$$\mathbf{E}_{\parallel k}=(V_k(z),0,W_k(z))\exp\left(i\frac{2\pi}{\lambda}n_1\sin\theta_1 x-i\omega t\right). \quad (106)$$

Here

$$W_k(z)=n_1\sin\theta_1 U_k(z)/\hat{\epsilon}_k=\mu_k n_1\sin\theta_1 U_k(z)/\hat{n}_k^2. \quad (107)$$

For the mean-square electric field, we have

$$\langle E_{\parallel kx}^2\rangle=\tfrac{1}{2}|V_k(z)|^2 \quad (108)$$

$$\langle E_{\parallel kz}^2\rangle=\tfrac{1}{2}|W_k(z)|^2 \quad (109)$$

and

$$\langle E_{\parallel k}^2\rangle=\tfrac{1}{2}(|V_k(z)|^2+|W_k(z)|^2). \quad (110)$$

The usual convention is to take the amplitude of the electric vector of the incident wave in phase 1 as unity. It should be noted when calculating $Q_k(z)$ for TM polarization that $H_1^{0t}=n_1E_1^{0t}=n_1$ for E_1^{0t} equal to unity.

DISCUSSION

The equations derived in this paper can be used to understand the physics of the interaction of radiation with a stratified medium in rather complex cases. They are, however, complicated, except for special cases, making it hard to visualize what results to expect. However, numerical calculations can readily be made by computer; when these are plotted as functions of various optical parameters, they clarify unknown optical behavior. We have used this procedure to develop a number of successful methods of studying optical properties.

The equations are true for any combination of absorbing and nonabsorbing films at any angle of incidence. For example, they hold true for frustrated-total-reflection interference-filter type calculations where the angle of incidence is greater than critical. However, the rules for choosing complex roots must be strictly followed. In the present paper, ξ_j must always be in the first quadrant. More subtle is the case of the phase angles, such as δ_1^r used in Eq. (63). An equation like Eq. (6), p. 625 of Born and Wolf[3] is not adequate for the determination of δ_1^r (B. and W.'s φ_{12}), because of ambiguities of signs of numerator and denominator. Correct results will be obtained using the equations of this paper if the signs as well as the magnitudes of phase angles are carefully noted.

Even though the multilayer media referred to above were assumed to comprise isotropic phases, some of the equations are valid for anisotropic phases. When dealing with TE polarization, for example, the only refractive index that matters in any phase is the one perpendicular to the plane of incidence. Also, for TM polarization at the critical angle, since the electric field in the final phase is perpendicular to the interphasal plane, the refractive index normal to the interphase boundary is the only one having an influence.

The general equations derived in this paper will be useful in ellipsometry, including ellipsometry in the total-reflection region beyond the critical angle. The accuracy of the approximate schemes now in use in ellipsometry can be checked by numerical comparison. The equations should also find wide application by those people studying light propagation in multilayer devices, such as laser-beam modulators.

SUMMARY

The main purpose of this paper is to derive explicit formulas for the mean-square electric fields induced by electromagnetic radiation in a two-phase, three-phase, and N-phase stratified medium. A number of other important formulas are also given, in general forms not found elsewhere. These include formulas for reflectance and transmittance. The mean-square electric field, rather than the field itself, is important. That is the quantity for which relatively simple relationships exist. No such simple relations exist for the electric field itself when absorbing phases are involved. Equations are first developed for the three-phase system. This system is simple enough, even when absorbing phases are involved, so that much physical insight can be had analytically from the equations in the form given. In the derivation, understanding of how the fields depend on optical parameters is emphasized. The two-phase case allows even greater physical insight, and simple equations result for special cases without loss of rigor. An example is the equations for fields at the critical angle. The equations for the N-layer case are derived in a different fashion, and while their physical interpretation is not simple, they are in a form easily programmed on a computer.

APPENDIX A

Explanation of Symbols Used

(1) Digit subscripts refer to the phase involved, counting from the incident phase as 1. When two numerical subscripts are used, two phases are involved. The subscript letters j and k indicate unspecified digits. If no subscript is used, the phase involved is to be understood from the text.

(2) Other subscripts: $\perp$ or $||$ indicate polarization (cf. below) and x, y, or z indicate vector component.

(3) Superscripts: 0 is used to indicate amplitudes of E and H fields and their components, t is used to indicate a wave traveling away from incident phase or toward final phase (transmitted), r indicates a wave in the opposite direction (reflected).

APPENDIX B

Choosing the Sign of Reξ_j and Imξ_j

If we represent a plane wave propagating in a homogeneous absorbing medium j by $\mathbf{E}_j=\mathbf{E}_j^0\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$, and choose the z axis as the direction of propagation, we have

$$\mathbf{E}_j=\mathbf{E}_j^0\exp\{i[(2\pi/\lambda)\xi_j z-\omega t]\}, \tag{B1}$$

where the quantity in parentheses is the variable phase. The phase could equally well be chosen as $(\omega t-(2\pi/\lambda)\xi_j z)$. However, the $\exp(-i\omega t)$ time factor of Eq. (B1) seems to be the most frequent choice. The mean-square field strength is given by

$$\tfrac{1}{2}(\mathbf{E}_j\cdot\mathbf{E}_j^*)=\tfrac{1}{2}|E_j^0|^2\exp\left[i\frac{2\pi}{\lambda}z(\xi_j-\xi_j^*)\right]$$
$$=\tfrac{1}{2}|E_j^0|^2\exp\left(-\frac{4\pi}{\lambda}\operatorname{Im}\xi_j z\right). \tag{B2}$$

The field strength must, of course, decrease with z in an absorbing medium. Therefore, for $\exp(-i\omega t)$ time dependence,

$$\operatorname{Im}\xi_j\geq 0 \text{ (time factor } e^{-i\omega t}). \tag{B3}$$

In a stratified plane-bounded system with transparent initial phase, for the jth phase

$$\xi_j\equiv\hat{n}_j\cos\theta_j=(\hat{n}_j^2-n_1^2\sin^2\theta_1)^{\frac{1}{2}}=(x+iy)^{\frac{1}{2}}, \tag{B4}$$

where x and y are the real and imaginary parts of the quantity under the radical. It is arbitrary whether $\hat{n}=n+in\kappa$, $\hat{n}=n-in\kappa$, $\hat{n}=n+ik$, or $\hat{n}=n-ik$ where n, κ, and k are positive real; and all four systems are used in the literature. The choice does, however, affect the root of ξ that must be taken, as will now be shown. In the four cases cited, if the sign of the last term is + then y is positive; if the sign is −, y is negative. The sign of x may be + or −. If y is + then $x+iy$ lies in the first two quadrants of the complex plane, and $\xi_j=(x+iy)^{\frac{1}{2}}$ lies in the first or third quadrants, depending on which root is taken. If Im$\xi_j\geq 0$, as given by (B3), the only choice possible is the first quadrant and

$$\operatorname{Re}\xi_j\geq 0,\ \operatorname{Im}\xi_j\geq 0 \text{ (time factor } e^{-i\omega t},\ \operatorname{Im}\hat{n}_j\geq 0). \tag{B5}$$

For this case, which is used by Stratton, Born and Wolf, and others, we have shown that $\delta_\perp^r$ and $\delta_{||}^r$ must be negative for a two-phase system. They are generally regarded as positive in the optics literature.

Peter Mueller (r., Tech Ops) at Holography III session, Detroit; E. N. Leith, chairman.