The Electric Susceptibility, Dielectric Constant, and Complex Index of Refraction

Electric Polarization: \( \mathbf{P}(\omega) = \varepsilon_0 \chi(\omega) \mathbf{E}(\omega) \)

Electric Displacement: \( \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \)

\[ \mathbf{D} = \varepsilon_0 (1 + \chi) \mathbf{E} = \varepsilon_0 \varepsilon_r(\omega) \mathbf{E} = \varepsilon \mathbf{E} \]

\( \chi(\omega) \) = complex frequency dependent electric susceptibility

\( \varepsilon_0 \) = permittivity of free space

\( \varepsilon \) = permittivity

\( \varepsilon_r(\omega) \) = relative permittivity or complex frequency dependent dielectric constant

\( \chi = \chi' + j\chi'' \)

\( \varepsilon_r = 1 + \chi = (1 + \chi') + j\chi'' \)

EM Plane Wave: \( \mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp(j \mathbf{k} \cdot \mathbf{r} - j \omega t) \)

In free space: \( k = \omega(\varepsilon_0 \mu_0)^{1/2} = \frac{\omega}{c} \)

\( c = (\varepsilon_0 \mu_0)^{-1/2} \)

\( c \) = speed of light

\( \mu_0 \) = permeability of free space

In a dielectric: \( k = \omega(\varepsilon_0 \mu_0)^{1/2} = \omega(\varepsilon_r \varepsilon_0 \mu_0)^{1/2} = \frac{n \omega}{c} \)

\( n = \varepsilon_r^{1/2} \)

\( n = \eta + j\kappa \)

\( \varepsilon_r = n^2 = (\eta^2 - \kappa^2) + j(2\eta\kappa) \)

\( n \) = complex index of refraction

\( \eta \) = (real) refractive index

\( \kappa \) = extinction coefficient

EM wave in z direction: \( E(z, t) = E_0 \exp\left( j \omega \left( \frac{\eta z}{c} - t - \frac{\omega \kappa z}{c} \right) \right) \)

\[ I(z) = I_0 \exp(-Kz) \]

Beer's Law:

\[ K = \frac{2\omega \kappa}{c} \]

\( K \) = Beer's Law absorption coefficient
Linear Electric Susceptibility Equations.

\[ \chi(\omega) = \frac{S\omega_0^2}{\omega_0^2 - \omega^2 + j\omega \Gamma} \]

\[ \text{Re}\chi = \frac{S\omega_0^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2} \]

\[ \text{Im}\chi = \frac{S\omega_0^2 \omega \Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2} \]

\[ \omega_0 = 100 \]

\[ \Gamma = 10 \]

\[ S = 1 \]
II. Classical Description of Electromagnetic Fields

From Maxwell's equations in free space, one can derive the electric and magnetic fields associated with a plane electromagnetic wave:

\[ E(r, t) = E_0 \exp\left(ik \cdot r - i\omega t\right) \]
\[ H(r, t) = H_0 \exp\left(ik \cdot r - i\omega t\right) \]

\(E\) and \(H\) are oscillating functions of time and space. The EM wave is propagating in the \(k\) direction. \(k\) is called the wavevector:

\[ |k| = \frac{\omega}{c} \quad \text{free space} \]

\[ k \cdot E = k \cdot H = 0 \]

\[ k \times E = \omega \mu_0 H \]

\[ k \times H = -\omega \varepsilon_0 E \]

\[ |H| = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} |E| \]

If \(k\) is in the \(z\) direction, then both \(E\) and \(H\) have the functional form:

\[ \exp\left[i\omega \left(\frac{k}{c} - z\right)\right] \]
When the EM wave is travelling through a dielectric medium, it induces a linear polarization \( P \):
\[
P = \varepsilon_0 \mathbf{X} \mathbf{E}
\]
where \( \mathbf{X} \), the proportionality constant, is called the electric susceptibility. \( \mathbf{X} \) is related to the wavevector \( \mathbf{k} \) by:
\[
\left( \frac{k \varepsilon_0}{\omega} \right)^2 = 1 + \mathbf{X}
\]
where \( \mathbf{X} \) is a complex quantity:
\[
\mathbf{X} = \mathbf{X}' + i \mathbf{X}''
\]

(Note: some define \( \mathbf{X} = \mathbf{X}' - i \mathbf{X}'' \))

The wavevector is also proportional to \( \eta \) and \( \mathbf{K} \):
\[
\frac{k \varepsilon_0}{\omega} = \eta + i \mathbf{K}
\]

\( \eta \) is the refractive index; \( \mathbf{K} \) is the extinction coefficient.

This is because the time and space varying parts of \( \mathbf{E} \) and \( \mathbf{H} \) now look like:
\[
\exp(ikz - \omega t) = \exp\left\{ i \omega \left( \frac{\eta z}{c} - t \right) - \frac{\omega k z}{c} \right\}
\]

Thus: \( \sqrt{\eta} \) is the velocity of the EM wave, and the last term \( \frac{\omega k z}{c} \) is an exponentially decaying function with \( z \).
The power of the EM wave is given by the Poynting vector averaged over one cycle of \( \omega \):

\[
I(t) = \frac{1}{2} \varepsilon_0 c \eta \left| \mathbf{E}(x,t) \right|^2
\]

\[
= \overline{I_0 \exp(-\beta x)}
\]

where \( \beta = \frac{2 \omega k}{c} \) is the Beer's law absorption coefficient.

Since

\[
(k_0 \omega)^2 = 1 + \chi = (\gamma + i \chi)^2
\]

then

\[
\gamma - \chi = 1 + \chi'
\]

and

\[
2 \chi k = \chi''
\]

Thus \( \chi' \) & \( \chi'' \) are related directly to the index of refraction & the extinction coefficient.

II. The Frequency-Dependent Susceptibility

\( \chi \) can in general be thought of as a function of \( \omega \). This can be described in terms of the four components of a general electric field \( E(t) \) (in lieu of the specific plane wave form of EM radiation):

\[
E(t) = \int_0^\infty E(\omega) \exp(-i \omega t) \, d\omega
\]
And, by the inverse FT:

\[ E(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t) \exp(i\omega t) \, dt \]

(since \( E(t) \) is real, \( E(-\omega) = E^*(\omega) \))

The polarization induced by \( E(t) \) is \( P(t) \):

\[ P(t) = \int_{-\infty}^{\infty} P(\omega) \exp(-i\omega t) \, d\omega. \]

And the frequency-dependent susceptibility is now defined as:

\[ P(\omega) = \varepsilon_0 \chi(\omega) E(\omega) \]

Because \( P(t) \) is real, we require that

\[ \chi(-\omega) = \chi^*(\omega) \]

which means that:

\[ \chi'(\omega) = \chi'(\omega) \]

\[ \chi''(\omega) = -\chi''(\omega) \]

These "causal relations" show that we only need \( \chi(\omega) \) for \( \omega > 0 \), for \( E(\omega) \exp(-i\omega t) \).

III. Classical Theory of the Susceptibility

Consider the system to be \( N \) damped harmonic oscillators at the resonant frequency \( \Omega_0 \):

\[ m \left( \ddot{x} + \Gamma \dot{x} + \omega_0^2 x \right) = -eE = -eE_0 \exp(-i\omega t) \]

The damping in this oscillator is the phenomenological damping that we included in the density matrix derivation.
We can solve this equation by seeking homogeneous solutions by plugging in the driving fields to get:

$$X(t) = \frac{-\varepsilon E(t)}{m(\omega_0^2 - \omega^2 - i \omega \Gamma)}$$

$$E(t) = E_0 \exp(-i\omega t)$$

The polarization induced in this collection of $N$ oscillators is given by:

$$P = N e X = \frac{N e^2 \varepsilon E_0 \exp(-i\omega t)}{m(\omega_0^2 - \omega^2 - i \omega \Gamma)}$$

The susceptibility $\chi(\omega)$ is therefore given by:

$$\chi(\omega) = \frac{N e^2 / \varepsilon}{\omega_0^2 - \omega^2 - i \omega \Gamma}$$

The form results in the same form for $\chi''$ and $\chi'$ as we derived quantitatively:

$$\chi'' = \frac{N e^2 \varepsilon^2}{m^2 \omega_0^4 \Gamma} \frac{\omega_0^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}$$

$$\chi' = \frac{N e^2 \varepsilon^2}{m^2 \omega_0^4 \Gamma} \frac{\omega_0^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}$$

If $\Gamma \ll \omega_0$, then $\omega_0$ is close to $\omega_0$, then

$$\chi'' \approx \frac{N e^2 \varepsilon^2 \omega_0 \Gamma}{m^2 (\omega_0^2 - \omega^2)^2 + \Gamma^2}$$

$\Gamma \ll \omega_0$.
This is once again the Lorentzian lineshape with width $\Gamma$:

In general, note that $\chi$ is distorted by the frequency-dependent refractive index $\eta$. ($\chi = \frac{\chi}{2\eta}$). This is sometimes called a dispersion effect. If $\eta$ doesn't change too much (i.e., when $\frac{\omega \eta}{m \Gamma} << 1$), then $\chi$ is a Lorentzian as well as $\chi''$.

As a final note, the $\chi''(\omega)$ and $\chi'(\omega)$ are related to each other by the "Kramers-Kronig relations":

$$\chi''(\omega) = -\frac{2\omega}{\pi} \Im \left\{ \frac{\chi'(\omega')}{\omega^2 - \omega'^2} \right\}$$

$$\chi'(\omega) = \frac{2}{\pi} \Re \left\{ \frac{\omega \chi''(\omega')}{\omega^2 - \omega'^2} \right\}$$

where $\Im$ denotes the principal part of the integral. For a derivation of these relations, see either Born & Wolf, Optics, or R. Loudon, *The Quantum Theory of Light*.